

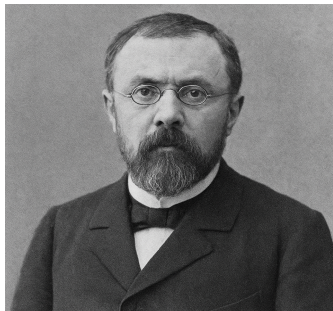
# *DIFFERENTIAL EQUATIONS IN MATHEMATICAL MODELING*

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# Differential Equations and the World



Henri Poincaré (1854–1912)

“The laws of nature are expressed by differential equations. To know the world is to understand these equations.”

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## Definition

A **pursuit trajectory** is the path followed by a pursuer (predator) that continuously adjusts its direction of motion in order to intercept a moving target.

**Mathematical model:** If the target moves along a trajectory  $T(t)$  and the pursuer's position is  $P(t)$ , then the pursuer's velocity vector  $\dot{P}(t)$  is always directed toward the instantaneous position of the target:

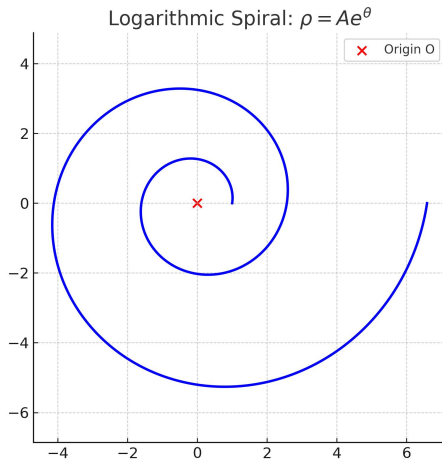
$$\dot{P}(t) = v \frac{T(t) - P(t)}{\|T(t) - P(t)\|}, \quad v = \text{constant speed of pursuer.}$$







This is the **logarithmic spiral**  $\rho = Ae^\theta$ ,  $A > 0$ .











Separating the variables:

$$\frac{dz}{\sqrt{z^2 + 1}} = \frac{a}{v} \frac{dy}{y}.$$

Integration gives:

$$\ln(z + \sqrt{z^2 + 1}) = \frac{a}{v} (\ln y + \ln C) = \ln(yC)^{a/v}.$$

Thus  $x'(y) = z(y) = \frac{1}{2} C_1 y^{a/v} - \frac{1}{2C_1} y^{-a/v}$  and by the initial position  $(x_0, y_0)$  of the drone we obtain

$$x'(y) = z(y) = \frac{1}{2} \left( \frac{y}{y_0} \right)^{a/v} - \frac{1}{2} \left( \frac{y}{y_0} \right)^{-a/v}$$





#### 4. *Conclusion*

- In the first problem, we obtain homogeneous ODE that can be reduced to equation with separable variables. The trajectory of the solution is a **logarithmic spiral**.
- In the second problem, the trajectory of the drone satisfies a nonlinear ODE of second order, resolvable via reduction of the order, obtaining equation with separable variables. The explicit formula of the solution is in terms of powers of  $y$ .
- Initial conditions fix the constants and give the precise path.















1. *Setup of the problem.*

Let  $y(t)$  denote the amount of salt in the tank at time  $t$ . The variation of  $y(t)$  is given by

$$dy(t) = [\text{inflow} - \text{outflow}] dt.$$

We observed only one instantaneous inflow of  $3 \text{ kg}$  at  $t = 5$ .

Hence

$$\text{inflow} = \begin{cases} 3 & t = 5, \\ 0 & \text{otherwise} \end{cases} = 3\delta_5(t),$$

where  $\delta_5$  is the Dirac delta centered at  $t = 5$ .

### 3. *The Dirac Delta Function.*

The Dirac delta  $\delta(t - c)$  is not a classical function but a *generalized function* (distribution) defined by its action on integrals:

$$\int_{-\infty}^{\infty} f(t) \delta(t - c) dt = f(c).$$

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In our problem,  $3\delta_5(t)$  represents the sudden addition of **3 kg** of salt at time  $t = 5$ .

#### 4. *Outflow and Differential Equation.*

The solution leaves the tank at  $1 \text{ L/min}$ . Since the concentration is  $y(t)/10$  (kg/L), the rate of salt leaving is

$$\frac{y(t)}{10}.$$

Thus, the governing equation is

$$y'(t) + \frac{y(t)}{10} = 3\delta_5(t), \quad y(0) = 5.$$



## 5. General Case with Impulse Input.

Consider the general differential equation

$$y' + ay = k\delta_c, \quad y(0) = y_0, \quad c \geq 0.$$

Applying the Laplace transform,

$$Y(s) = \frac{y_0}{s+a} + k \frac{e^{-cs}}{s+a}.$$

Recall that

$$\mathcal{L}(y)(s) = Y(s) = \int_0^{+\infty} y(t)e^{-st} dt$$

is *Laplace transform* of the function  $f(t)$ ,  $t \in (0, +\infty)$ .



The inverse Laplace transform gives

$$y(t) = \begin{cases} y_0 e^{-at}, & 0 \leq t < c, \\ y_0 e^{-at} + k e^{-a(t-c)}, & t \geq c. \end{cases}$$

**Observation:** The solution is discontinuous at  $t = c$  because of the instantaneous impulse input.





### 1. *System Reformulation.*

Let  $T_g(t) = x(t)$ ,  $T_b(t) = y(t)$ , and  $T_e = r(t)$ . Then the system becomes

$$\begin{cases} \dot{x} = a(y - x), \\ \dot{y} = a(x - y) + b(r(t) - y). \end{cases}$$

Or equivalently:

$$\begin{cases} \dot{x} = -ax + ay, \\ \dot{y} = ax - (a + b)y + br(t). \end{cases}$$

If  $r(t) = 0$ , the system is homogeneous.







## Three-Body Model

Consider a thermally isolated system consisting of three bodies that exchange heat.

By Newton's law of cooling, if  $x_1(t)$ ,  $x_2(t)$ ,  $x_3(t)$  denote their temperatures at time  $t$ , then:

$$\begin{cases} \dot{x}_1 = a(x_3 - x_1) + a(x_2 - x_1), \\ \dot{x}_2 = a(x_1 - x_2) + a(x_3 - x_2), \\ \dot{x}_3 = a(x_1 - x_3) + a(x_2 - x_3). \end{cases}$$

*Simplified Case.*

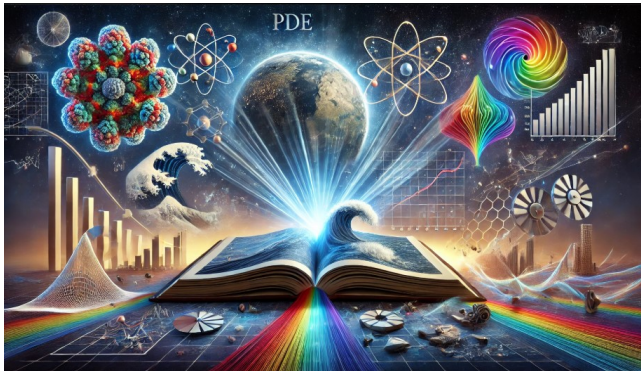
Let  $\mathbf{a} = \mathbf{1}$ . Then, in matrix form:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

The general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{0t} + C_2 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} e^{-3t} + C_3 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^{-3t}.$$

PDEs are fundamental mathematical tools to model phenomena where variables depend on both time and space, as happens in many physical, biological, economic, and engineering systems.





### *Smoothing Effect of the Heat Equation:*

- A fundamental property of the heat equation is its **smoothing effect**.
- Solutions become smoother over time, regardless of the irregularity of initial data.
- Even if the initial distribution is discontinuous, it quickly smooths out and tends towards equilibrium.
- This phenomenon is crucial not only in physics, but also in applied sciences where stability and equilibrium are important.

We consider a solid material in  $\mathbb{R}^3$ ,  $\mathbf{x} = (x, y, z)$ . Denote by

- $q(\mathbf{x}, t)$  *heat current density*;
- $K$  *thermal conductivity*;
- $q \cdot \mathbf{n}$  *the heat flux* in the direction  $\mathbf{n}$ .  
It measures the rate of heat flow per unit time per unit area across a plane with normal vector  $\mathbf{n}$ .
- $r(\mathbf{x}, t)$  *internal heat source*.

We consider a solid material occupying a region of three-dimensional space. **Fourier's law:**

$$\mathbf{q} = -k\nabla u,$$

- $\nabla u$  points in the direction of the maximum increase of temperature  $u$ .
- The negative sign reflects that heat flows from warmer to cooler regions.
- $\mathbf{q}$  therefore points in the direction of maximum *decrease* of  $u$ , and  $|\mathbf{q}|$  is the rate of heat flow in that direction.

During a small time interval  $(t, t + \Delta t)$ , heat flows through the material and may also be generated by internal sources at a rate  $s(x, t)$ .

Therefore, the amount of heat that enters in any region  $R$  of the material within the time interval  $(t, t + \Delta t)$  is, to first order in  $\Delta t$ , given by

$$Q = \left( -\int_{\partial R} \mathbf{q} \cdot \mathbf{n} dS + \int_R s dV \right) \Delta t + O((\Delta t)^2),$$

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The same heat  $Q$  increases the temperature:

$$Q = \int_R c\rho u_t dV \Delta t + O((\Delta t)^2),$$



Consider the heat equation

$$u_t = K\Delta u + r(\mathbf{x}, t) \quad \mathbf{x} \in D = \mathbb{R} \times \mathbb{R} \times (0, L), \quad t > 0.$$

- In the next we will make a detailed study of the heat equation in a slab, defined by

$$\Omega = \left\{ (x, y) \in \mathbb{R}^2, \quad 0 < z < L \right\}.$$

This mathematical model is appropriate for a wall of thickness  $L$ , where we ignore the variations of temperature in the  $x, y$  directions.

- The boundary conditions at the surfaces  $z = 0$  and  $z = L$  reflect the thermal properties of the inside (resp. outside) of the wall.

## Steady-state solutions

An important class of solutions of the heat equation are the *steady-state* solutions. This means that  $\frac{\partial u}{\partial t} = 0$ .

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Calculate SSS of

$$\begin{cases} u_t = K\Delta u & \mathbf{x} \in \Omega \\ u(x, y, 0) = T_1, & (\partial u / \partial z + hu)(x, y, L) = 0, \quad h, T_1 > 0. \end{cases}$$

Compute the flux through the faces of the slab.

**Solution.** We look for  $u(x, y, z) = U(z)$ . Thus  $U(z) = A + Bz$ .



Let us compute the flux through the faces of the slab, that is

$$\Phi_{\mathbf{n}} = -K\nabla u \cdot \mathbf{n}$$

$$\mathbf{n}_L(x, y, L) = (0, 0, 1), \quad \mathbf{n}_0(x, y, 0) = (0, 0, -1)$$

Thus the flux from the upper face is

$$\Phi_{\mathbf{n}_L} = -KU'(z) = KhT_1/(1 + hL),$$

while the flux from the lower face is

$$\Phi_{\mathbf{n}_0} = KU'(z) = -KhT_1/(1 + hL).$$

□



We consider a flat surface. The temperature on the surface  $z = 0$  is independent of location and depends only on time.

$$\begin{cases} u_t = Ku_{zz} & z > 0, \quad -\infty < t < +\infty \\ u(0, t) = u_0(t), & -\infty < t < +\infty \\ u_0(t + \tau) = u_0(t) \end{cases}$$

In addition we require

$$|u(z, t)| \leq M.$$

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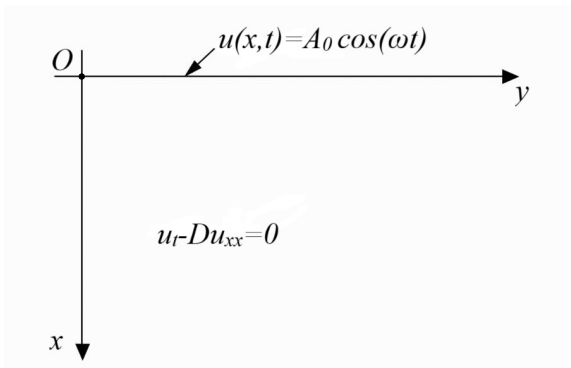
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We are interested of complex solutions  $u(z, t) = Z(z)T(t)$ , hence

$$\frac{KZ''(z)}{Z(z)} = \frac{T'(t)}{T(t)} = -\lambda \in \mathbb{C}.$$



- The model describes periodic oscillations at the surface ( $x = 0$ ).
- We want to understand how these oscillations propagate with depth.

We get the ODEs

$$T'(t) + \lambda T(t) = 0, \quad T(t) = e^{-\lambda t}$$

$$Z''(z) + \frac{\lambda}{K} Z(z) = 0, \quad Z(z) = e^{(-1+i)z\sqrt{\beta/2K}}.$$

Since we require *bounded solution*,  $\lambda = i\beta$ ,  $\beta \in \mathbb{R}$ . Per simplicity take  $\beta > 0$ , if it is negative no new solutions are obtained.

The *complex separated solutions* are

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Taking the real and imaginary parts, we have the real *quasi-separated solutions*

$$e^{-cz} \cos(\beta t - cz), \quad e^{-cz} \sin(\beta t - cz), \quad c = \sqrt{\beta/2K}.$$

Any linear combination is a solution too.



To solve the original problem we take *the Fourier expansion of the boundary function*

$$u_0(t) = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{2n\pi t}{\tau} + B_n \sin \frac{2n\pi t}{\tau} \right).$$

Since  $u(0, t) = u_0(t)$  this holds if

$$\beta_n = 2\pi n/\tau, \quad c_n = \sqrt{n\pi/K\tau}$$

in the *quasi-separated solutions*. Hence

$$u(z, t) = A_0 + \sum_{n=1}^{\infty} e^{-c_n z} \left[ A_n \cos(\beta_n t - c_n z) + B_n \sin(\beta_n t - c_n z) \right].$$

Find the periodic solution of the problem

$$\begin{cases} u_t = Ku_{zz} & z > 0, -\infty < t < +\infty \\ u(0, t) = A_0 + A_1 \cos \frac{2\pi t}{T}, & -\infty < t < +\infty \\ A_0, A_1, T > 0. \end{cases}$$

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**Solution.**

We note that

$$u_0(t + T) = u_0(t), \quad \text{the solution is } T\text{-periodic.}$$

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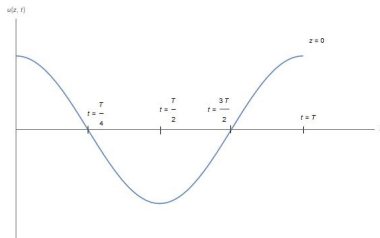
Referring to the general solution obtained before we write

$$A_0 + A_1 \cos \frac{2\pi t}{T} = A_0 + \sum_{n=1}^{\infty} \left( A_n \cos \frac{2n\pi t}{T} + B_n \sin \frac{2n\pi t}{T} \right)$$



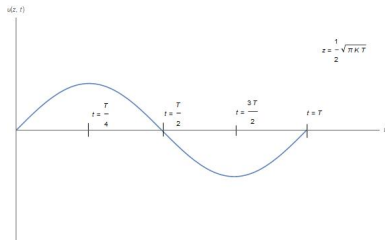
## Numerical example

Graph of the solution as a function of  $t \in [0, T]$  :



$$z = 0$$

$$u(0, 0) = A_0 + A_1$$



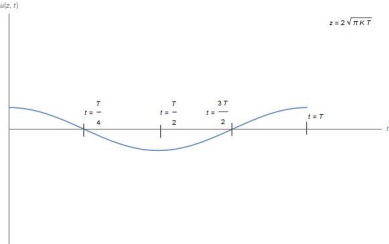
$$z \sqrt{\pi / K T} = \frac{\pi}{2}$$

$$u(z, 0) = A_0$$



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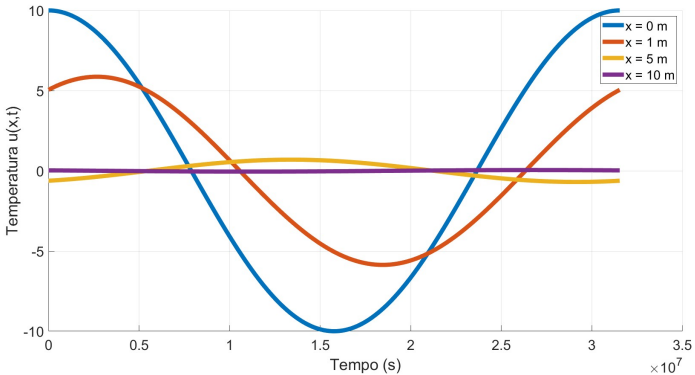
$$z\sqrt{\pi/KT} = 2\pi,$$

$$u(z, 0) = A_0 + A_1 e^{-2\pi}$$





# TEMPERATURE PROFILES



- Amplitude decreases with depth.
- Phase shift increases with depth.
- Deep soil temperature is more stable and delayed compared to the surface.



Consider a stretch  $[c, d] \subset [a, b]$ .

We ask: *How does the number of vehicles in  $[c, d]$  change over  $[t, t + \epsilon]$ ?*

$$(5) \quad Q(\epsilon) = \int_c^d [g(x, t + \epsilon) - g(x, t)] dx$$

Instantaneous variation of the number of vehicles in  $[c, d]$  is

$$(6) \quad \ell = \lim_{\epsilon \rightarrow 0} \frac{Q(\epsilon)}{\epsilon}$$









Solution:

$$g(x, t) = \begin{cases} g_{max}, & x \leq -v_{max}t \\ \frac{g_{max}}{2} \left( 1 - \frac{x}{v_{max}t} \right), & -v_{max}t < x < v_{max}t \\ 0, & x \geq v_{max}t \end{cases}$$

## Rarefaction wave centered at origin

